Exercise 42

Use logarithmic differentiation to find the derivative of the function.

$$y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned}
\ln y &= \ln\left[\sqrt{x}\, e^{x^2-x}(x+1)^{2/3}\right] \\
&= \ln\sqrt{x} + \ln e^{x^2-x} + \ln(x+1)^{2/3} \\
&= \ln x^{1/2} + (x^2-x)\ln e + \frac{2}{3}\ln(x+1) \\
&= \frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)
\end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left[\frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)\right] \\
\frac{1}{y}\cdot\frac{d}{dx}(y) &= \frac{1}{2}\left(\frac{1}{x}\right) + 2x - 1 + \frac{2}{3}\left(\frac{1}{x+1}\right)\cdot\frac{d}{dx}(x+1) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\cdot(1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3(x+1) + 3(x+1)(2x)(2x-1) + 2(2x)}{6x(x+1)} \\
\frac{dy}{dx} &= y\left[\frac{12x^3 + 6x^2 + x + 3}{6x(x+1)}\right] \\
&= \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}\left[\frac{12x^3 + 6x^2 + x + 3}{6x(x+1)}\right] \\
&= \frac{e^{x^2-x}}{6\sqrt{x}\,(x+1)^{1/3}}(12x^3 + 6x^2 + x + 3)
\end{aligned}$$